Εξίσωση Navier-Stokes

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\vec{\nabla}P - \rho\vec{\nabla}h + \nu\nabla^2\vec{u} \quad , \nu = \frac{\mu}{\rho}$$

επιτάχυνση = δύναμη πίεσης + βαρύτητας + ιξώδους

Ερπυστική ροή $\left(\frac{D\vec{u}}{Dt} = 0\right)$

μόνιμη μικρή $\vec{u}$

$$\cancel{\frac{\partial \vec{u}}{\partial t}} + \cancel{(\vec{u}\cdot\vec{\nabla})\vec{u}} = 0$$

$$\begin{cases} 0 = -\vec{\nabla}P + \mu\nabla^2\vec{u} \\ \vec{\nabla}\cdot\vec{u} = 0 \end{cases} \rightarrow \begin{cases} \vec{\nabla}\cdot \Rightarrow \nabla^2 P = 0 \\ \vec{\nabla}\times \Rightarrow \nabla^2 \vec{\zeta} = 0 \end{cases}$$

$$\vec{\nabla}\times\{-\vec{\nabla}P + \mu\nabla^2\vec{u}\} = \mu\vec{\nabla}\times(\nabla^2\vec{u}) = \mu\nabla^2(\vec{\nabla}\times\vec{u}) = \mu\nabla^2\vec{\zeta}$$

2-διάστατη ροή $\Rightarrow \psi(x,y)$

$$\nabla^2\psi(x,y) = \zeta_z \qquad \vec{\zeta} = (0,0,\zeta_z)$$

$$\frac{\partial\psi}{\partial y} = u_x \qquad \frac{\partial\psi}{\partial x} = -u_y$$

$$\nabla^2\zeta_z = \nabla^4\psi = 0 \Rightarrow \frac{\partial^4\psi}{\partial x^4} + 2\frac{\partial^4\psi}{\partial x^2\partial y^2} + \frac{\partial^4\psi}{\partial y^4} = 0$$

διαρμονική εξίσωση

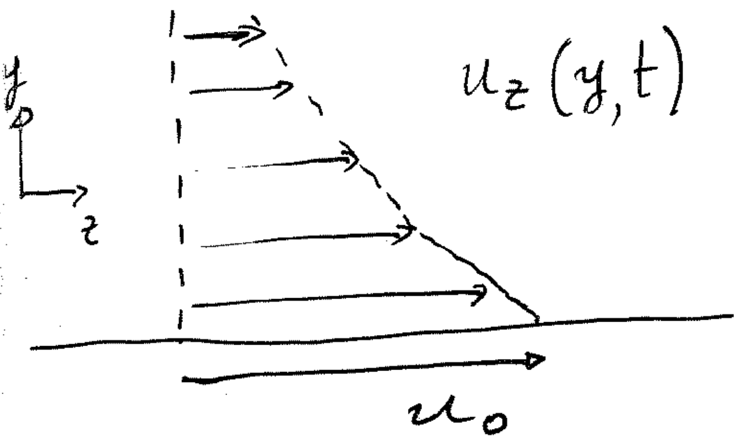
Διάχυση στην εξίσωση Navier-Stokes

Ροή μη μόνιμη με χαμηλή $\underset{\vec{u}}{\vec{\nabla} P}$, Re

$$\frac{\partial \vec{u}}{\partial t} = \nu \nabla^2 \vec{u}$$ εξίσωση διάχυσης $D_0 = \nu$

Τί διαχέεται;

Κινούμενη Πλάκα



$t = 0 \quad u_z(y,t) = 0$

$t > 0 \quad u_z(0,t) = u_0$

$u_z(\infty, t) \to 0$

$$\frac{\partial u_z}{\partial t} = \nu \frac{\partial^2 u_z}{\partial y^2}$$

απομάκρυνση τυχαίου βηματισμού $y = \sqrt{D_0 t}$

$$\Rightarrow u_z(y,t) = f(\eta) \qquad \eta = \frac{y}{2\sqrt{D_0 t}}$$

$$2\eta \frac{df}{d\eta} + \frac{d^2 f}{d\eta^2} = 0 \quad \xrightarrow{\text{ολοκλήρωση}} \quad \frac{df}{d\eta} = c_1 e^{-\eta^2}$$

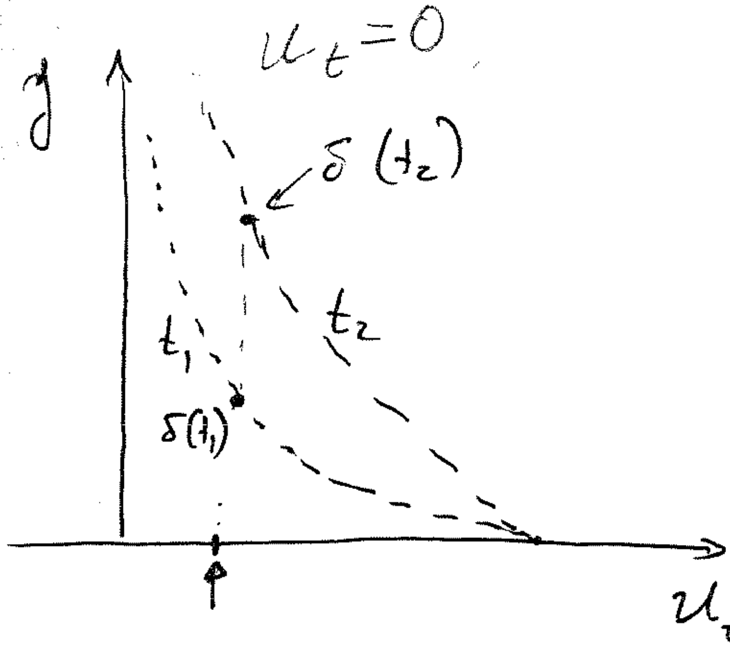
$$\frac{d^2 f / d\eta^2}{df/d\eta} = -2\eta$$

$$u_z(\eta) = c_2 + c_1 \int_0^\eta e^{-\eta'^2} d\eta'$$

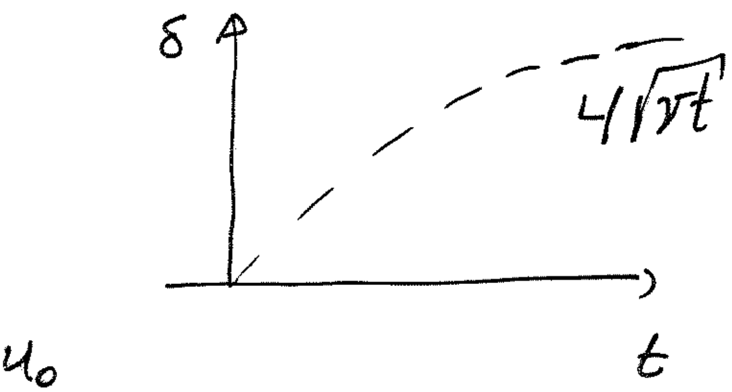
$u_z(0,t) = u_0 \Rightarrow c_2 = u_0$

$u_z(\infty,t) = 0 \Rightarrow c_1 = -u_0$

$$u_z(\eta) = u_0 \left[1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta'^2} d\eta'\right] = u_0 \left[1 - \mathrm{erf}(\eta)\right]$$



$$\frac{\delta}{2\sqrt{\nu t}} = 2 \Rightarrow u_z \simeq 0.01 u_0$$



αέρα σε $20°C$ $\quad \nu = 1.5 \times 10^{-5}\, m^2/sec$

$\Rightarrow \delta = 11\, cm$ σε $t = 1$ λεπτό

νερό $\quad \nu = 1 \times 10^{-6}\, m^2/sec \Rightarrow \delta = 3\, cm$

Για μεταπόπιση πλάκας κατά L σε $t = \frac{L}{u_0}$ έχουμε

$$\delta = 4\sqrt{\nu \frac{L}{u_0}} \rightarrow 4L\sqrt{\frac{\nu}{u_0 L}} = \frac{4L}{\sqrt{Re}}$$

Στροβιλισμός $\sim \zeta_x \sim \frac{\partial u_z}{\partial y} \quad \frac{4u_0}{\sqrt{\pi \nu t}} e^{-\frac{y^2}{4\nu t}}$

Περιοχή στροβιλισμού $\sim \delta(t)$

Ιξωδική απόσβεση γραμμικού στροβίλου

$$\vec{u} = \frac{\Gamma_0}{2\pi R}\,\hat{e}_\phi \quad \text{χωρίς ιξώδες}$$

$$\hat{e}_r \to -\frac{u_\phi^2}{R} = -\frac{1}{\rho}\frac{\partial P}{\partial R} \Rightarrow P(R,t) \qquad \mu \neq 0$$

$$\hat{e}_\phi \to \frac{\partial u_\phi}{\partial t} = \nu\left(\frac{\partial^2 u_\phi}{\partial R^2} + \frac{1}{R}\frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R^2}\right)$$

$$\Gamma(R,t) = 2\pi R\, u_\phi(R,t) \quad \text{– κυκλοφορία}$$

$$\frac{\partial \Gamma}{\partial t} = \nu\left(\frac{\partial^2 \Gamma}{\partial R^2} - \frac{1}{R}\frac{\partial \Gamma}{\partial R}\right), \qquad \Gamma(R,0) = \Gamma_0, \quad \Gamma(0,t) = 0 \;(t$$

$$\Gamma(R,t) = f(\eta) \qquad \eta = \frac{R}{\sqrt{\nu t}}$$

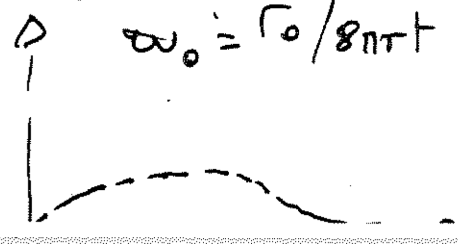
$$\Gamma(R,t) = \Gamma_0\left(1 - e^{-R^2/\nu t}\right) \Rightarrow \Gamma_0$$

$$u_\phi(R,t) = \frac{\Gamma_0}{2\pi R}\left(1 - e^{-R^2/\nu t}\right) \xrightarrow{R \gg \sqrt{\nu t}} \frac{\Gamma_0}{2\pi R}$$

$$R \gg \sqrt{\nu t} \qquad u_\phi = \frac{\Gamma_0}{2\pi R} \qquad P(R,t) = -\left(\frac{\Gamma_0}{2\pi}\right)^2 \frac{1}{2r^2}$$

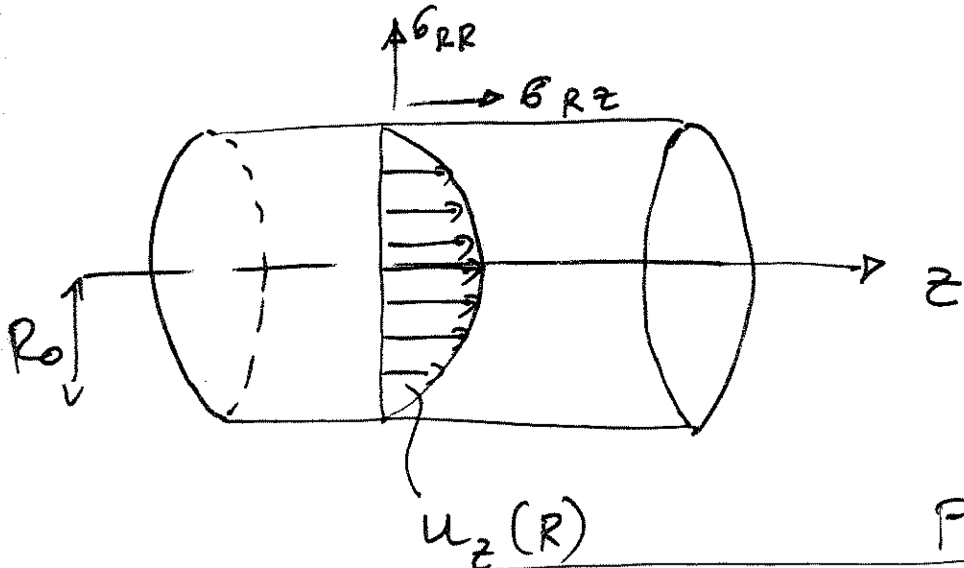
$$R \ll \sqrt{\nu t} \qquad u_\phi \simeq \frac{\Gamma_0 R}{8\pi\nu t} = \omega_0 R \qquad = P_0 + \frac{1}{2}\frac{R^2}{\omega_0^2}$$

$$\omega_0 = \Gamma_0 / 8\pi\nu t$$



Στρωτή ροή Poiseuille

Ιξώδης ροή χωρίς αδράνεια $\quad \left.\begin{array}{l}\vec{u}\cdot\vec{\nabla}\vec{u}=0\\ \simeq 0\end{array}\right\}.$



ανεξάρτητο του ϕ

με $\dfrac{\partial P}{\partial z} = C$

P ανεξάρτητο R, ϕ

χωρίς βαρύτητα

$$\boxed{0 = -\frac{dP}{dz} + \mu \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial u_z}{\partial R}\right)}$$

$$u_z(R=R_0) = 0$$

$$\frac{du_z}{dR}(R=0) = 0$$

$$2\pi R\,\sigma_{Rz} = 2\pi R\mu\frac{\partial u_z}{\partial R}$$

δύναμη ανά μονάδα μήκους

Δύναμη σε κυλινδρικό δακτύλιο πάχους dR

ιξώδους

$$\left(2\pi R\mu\frac{\partial u_z}{\partial R}\right)_{R+\frac{dR}{2}} - \left(2\pi R\mu\frac{\partial u_z}{\partial R}\right)_{R-\frac{dR}{2}} = 2\pi\mu\frac{\partial}{\partial R}\left(R\frac{\partial u_z}{\partial R}\right)$$

βαθμίδα πίεσης

$$-2\pi R\,dR\,\frac{\partial P}{\partial z}.$$

$$\frac{1}{2} R^2 \frac{dP}{dz} = \mu R \frac{du_z}{dR} + C_1 \Rightarrow C_1 = 0, \quad \frac{\partial u_z}{\partial R}(R=0)=0$$

$$\Downarrow$$

$$\frac{R^2}{4}\frac{dP}{dz} = \mu u_z + C_2 \Rightarrow C_2 = \frac{R_0^2}{4}\frac{dP}{dz}, \quad u_z(R_0)=0$$

$$u_z = -\frac{R_0^2}{4\mu}\frac{dP}{dz}\left(1-\frac{R^2}{R_0^2}\right) = u_z(0)\left(1-\frac{R^2}{R_0^2}\right)$$

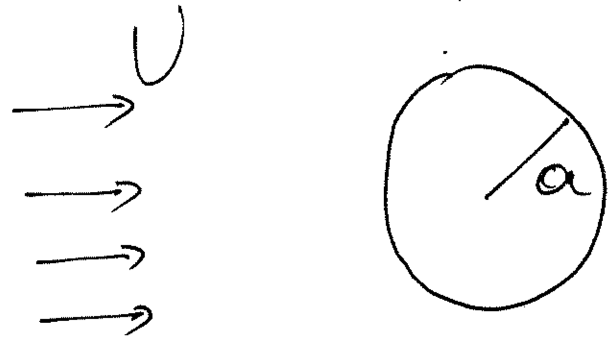
$$\vec{\zeta} = \vec{\nabla} \times \vec{u} = 2u_z(0)\frac{R}{R_0^2}\hat{e}_\phi$$

$Q = \bar{u}_z \pi R_0^2 -$ u_z - μέση ροή ταχύτητας

$$Q = \int_0^{R_0} u_z 2\pi R \, dR = -\frac{\pi R_0^4}{8\mu}\frac{dP}{dz}.$$

$$\bar{u}_z = \frac{R_0^2}{8\mu}\frac{dP}{dz} = \frac{1}{2}u_z(R=0)$$

Ερπυστική ροή γύρω από σφαίρα



1η μέθοδος

$$\textcircled{1}\ \vec{\nabla}\cdot\vec{u}=0$$
$$\textcircled{2}\ -\vec{\nabla}p+\mu\nabla^2\vec{u}=0$$

$$\vec{\nabla}\cdot\textcircled{2}\rightarrow\ \boxed{\begin{matrix}\nabla^2 p=0\ \text{Laplace}\\ -\vec{\nabla}p+\mu\nabla^2\vec{u}=0\end{matrix}}$$

Οριακές συνθήκες

$u_r(a,\theta)=u_\theta(a,\theta)=0$

$\vec{u}(\infty,\theta)=U\hat{i}$

$$\vec{\nabla}\times\textcircled{2}\Rightarrow\nabla^2\vec{\zeta}=0\xrightarrow{2D}\nabla^2\psi=\zeta_z$$

$$\Rightarrow\boxed{\begin{matrix}\nabla^4\psi=0\\ u_r=\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\\ u_\theta=-\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\end{matrix}}\leftarrow\vec{\nabla}\cdot\vec{u}=0$$

2η μέθοδος

$\nabla^2 p=0\Rightarrow$ Γραμμικός συνδιασμός πηγών πίεσης

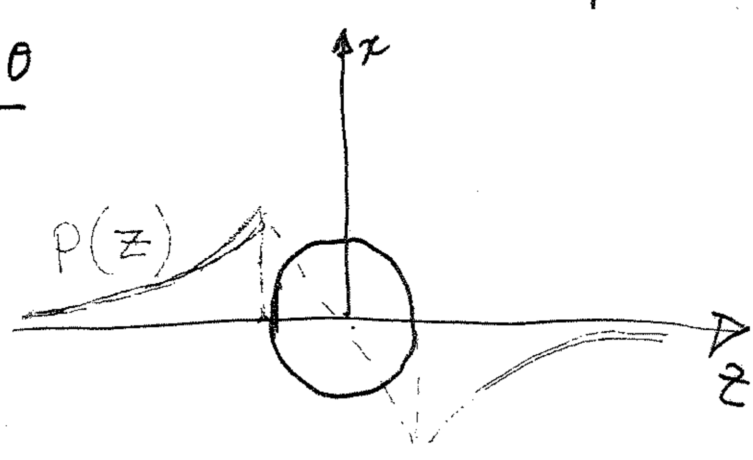
$p(r,\theta)$ – $r\to\infty$ $p\to0$ πώς $p\sim\frac{1}{r^n}$??

$n\leq3$ $p_{\text{δυναμ}}\sim\frac{1}{r^3}$

Μια λύση

$$p(r,\theta)=-\frac{Az}{r^3}=-\frac{A\cos\theta}{r^2}$$

$$p(\pi-\theta)=-p(\theta)$$



$\nabla^2\vec{u}=A\vec{\nabla}(\cos\theta)$

Συνάρτηση Ροής Stokes ψ

$$\nabla^4\psi = 0 \implies \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}\right)^2\psi = 0$$

Με οριακές συνθήκες

$u_r(a,\theta) = 0$ $\qquad r = a: \qquad \dfrac{\partial\psi}{\partial r} = \dfrac{\partial\psi}{\partial\theta} = 0$

$u_\theta(a,\theta) = 0$ $\qquad ;\ r \to \infty: \qquad \psi \to \tfrac{1}{2}Ur^2\sin^2\theta + \text{const}$

$\vec{u}(\infty,\theta) \to U\hat{i}$

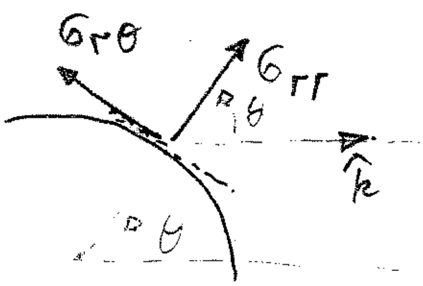
σταθερή ροή

$$\psi(r,\theta) = f(r)\sin^2\theta \implies \psi = \frac{1}{4}Ua^2\sin^2\theta\left(\frac{a}{r} - \frac{3r}{a} + \frac{2r^2}{a^2}\right)$$

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)^2 f = 0 \qquad f \sim r^s \qquad s = -1, 1, 2$$

$s = 4$ απορρίπτεται

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \implies u_r = U\cos\theta\left(1 + \frac{a^3}{2r^3} - \frac{3a}{2r}\right), \quad u_\theta = U\sin\theta\left(-1 + \frac{a^3}{4r^3} + \frac{3a}{4r}\right)$$



$$p = p_\infty - \frac{3\mu aU}{2r^2}\cos\theta$$

$$\tau_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = -\frac{\mu U\sin\theta}{r}\left(\frac{3a^3}{2r^3}\right)$$

$$F = -\int_0^\pi \tau_{r\theta}\Big|_{r=a}\sin\theta\, dA - \int_0^\pi p\Big|_{r=a}\cos\theta\, dA$$

$$dA = 2\pi a^2\sin\theta\, d\theta$$

$$\boxed{F = 4\pi\mu Ua + 2\pi\mu Ua = 6\pi\mu Ua}$$

Δύναμη στην x-και ανά μονάδα επιφάνειας

$$\sigma_x = \left(\sigma_{rr}\Big|_{r=a} \overset{\cos\theta}{\hat{e}_r} + \sigma_{r\theta}\Big|_{r=a} \overset{-\sin\theta}{\hat{e}_\theta}\right) \cdot \hat{k}$$

$$\sigma_{rr} = -P + 2\mu \frac{\partial u_r}{\partial r} = -P + 2\mu \cos\theta \cdot \left(-\frac{3}{2}\frac{}{r^4} + \frac{3a}{2r^2}\right)$$

$$= \mu a U \cos\theta \left\{\frac{3}{2}\frac{1}{r^2} - \frac{}{2\,r^4}\right\}$$

$$\sigma_{r\theta} = -\frac{\mu U \sin\theta}{r^4}\frac{3}{2}a^3$$

$$\sigma_x\Big|_{r=a} = \frac{3\mu U}{2a}\cos^2\theta + \frac{3}{2}\frac{\mu U}{a}\sin^2\theta$$

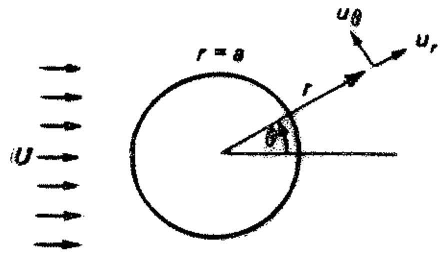
$$= \frac{3}{2}\frac{\mu U}{a}$$

$$\boxed{F_D = 4\pi a^2 \sigma_x\Big|_{r=a} = 6\pi\mu a U}$$

ανεξάρτητο Re

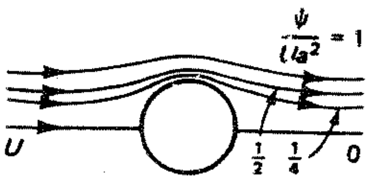
Συντελεστής αντίστασης

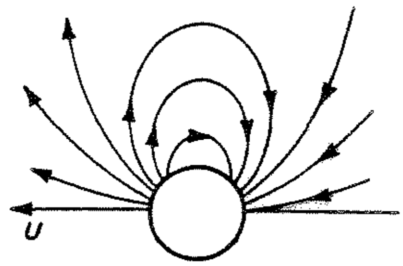
$$C_D = \frac{F_D}{\quad} = \frac{6\pi\mu a U}{\frac{1}{2}\rho U^2(\pi a^2)} = \frac{24}{Re} \qquad \left(Re = \frac{U}{\nu}\right.$$



Coordinate system

$\mu = 0$



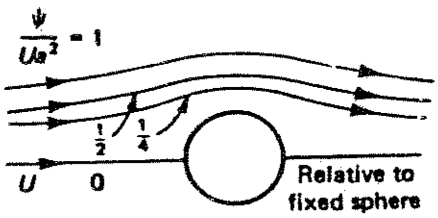


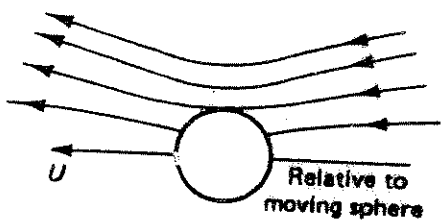
Potential flow

ροή γύρω από σφαίρα

$\mu \neq 0$

κινούμενη σφαίρα





Stokes flow

$$\psi = \tfrac{3}{2}\nu r_0(1 + \cos\theta)\left[1 - \exp\left(\frac{-U_\infty r(1 - \cos\theta)}{2\nu}\right)\right] - \frac{1}{4}\frac{U_\infty r_0^3}{r}\sin^2\theta,$$

πρόβλημα ω $r \to \infty$

$(\vec{u}\cdot\vec{\nabla})\vec{u} \simeq U\dfrac{\partial \vec{u}}{\partial z}$

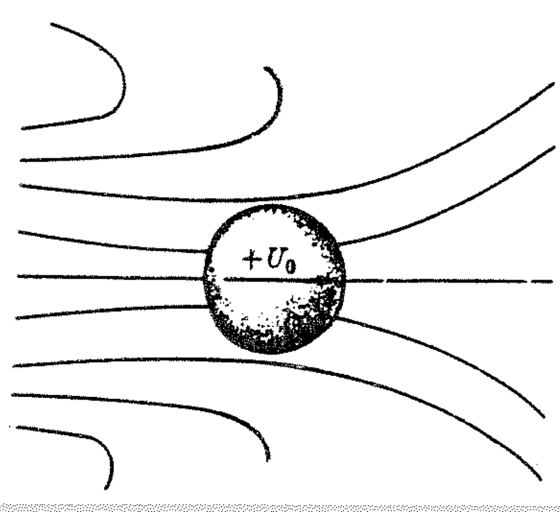
λύση

Oseen

$\vec{u}\cdot\vec{\nabla}) \sim U_0 \frac{u}{r^2}$

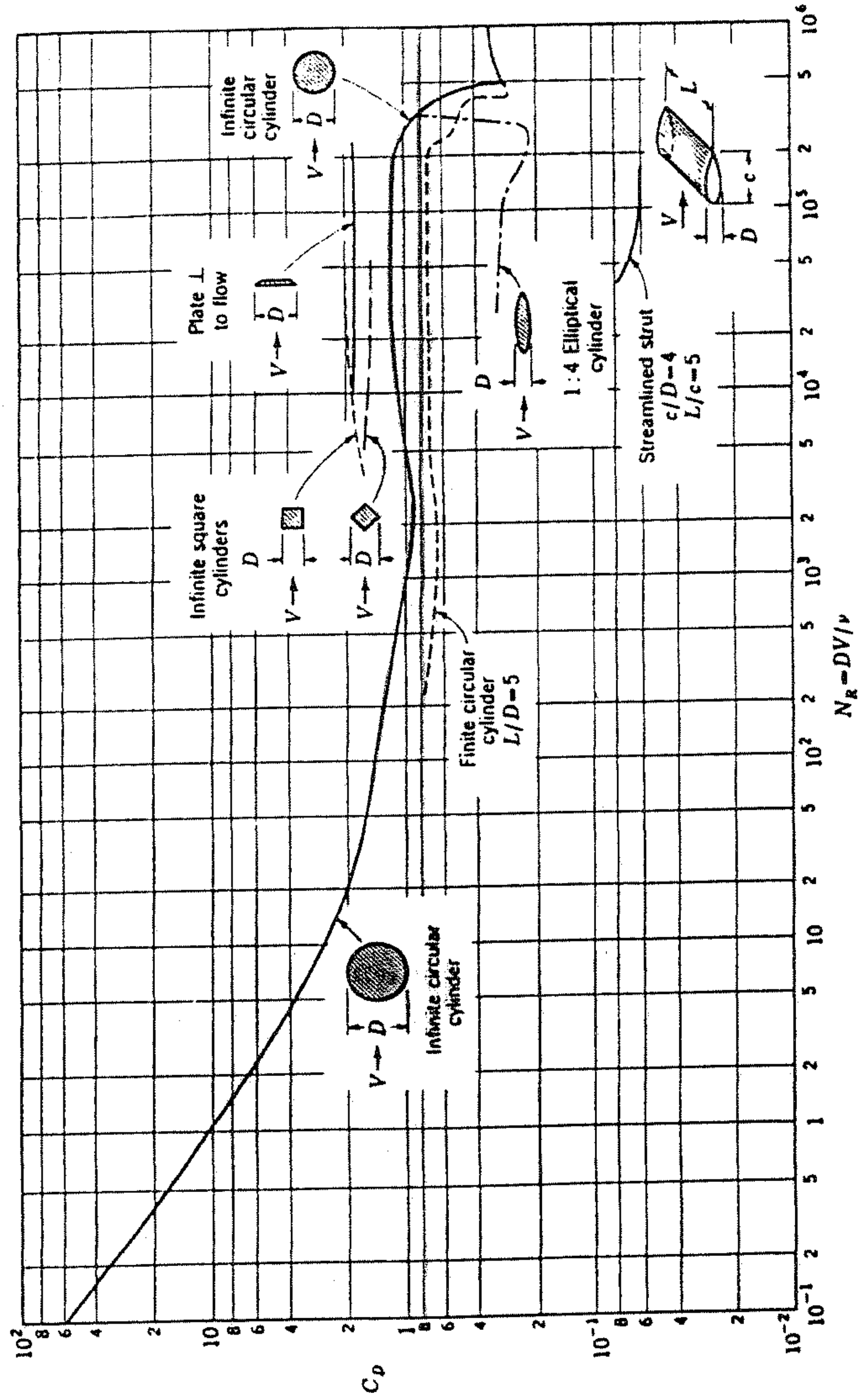
$\sim \nu u \frac{1}{r^3}$

$Re\ r$



$C_D = \dfrac{24}{Re}\left(1 + \dfrac{3}{16}Re\right)$

Κυλινδρικά σώματα

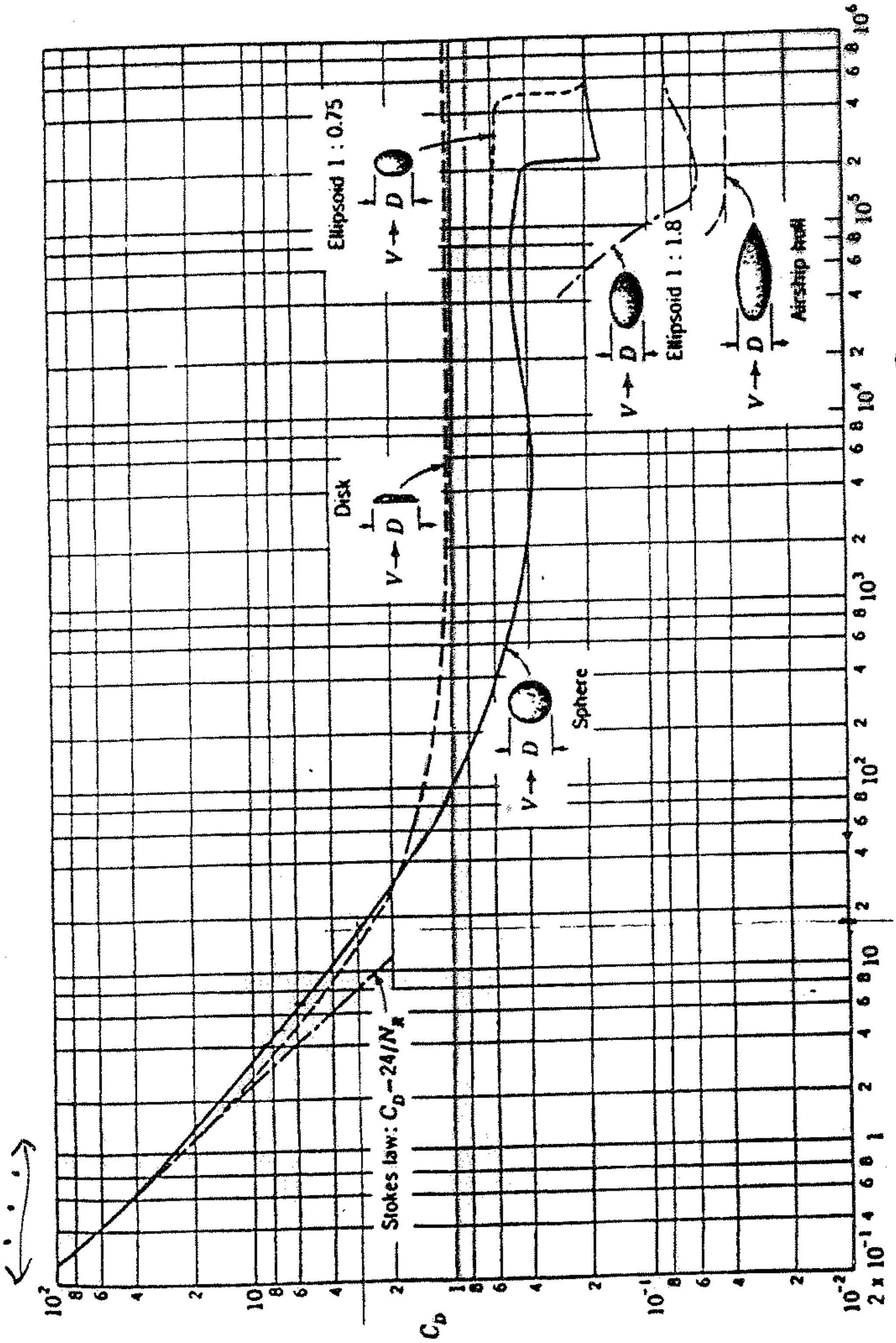


Drag coefficients for two dimensional bodies as a function of Reynolds Number (N_R)

(from B.S. Massey, Mechanics of Fluids, 6th ed. Van Nostrand Reinhold, 1989)

3-D σώματα

$Re < 1$



Re

Reynolds

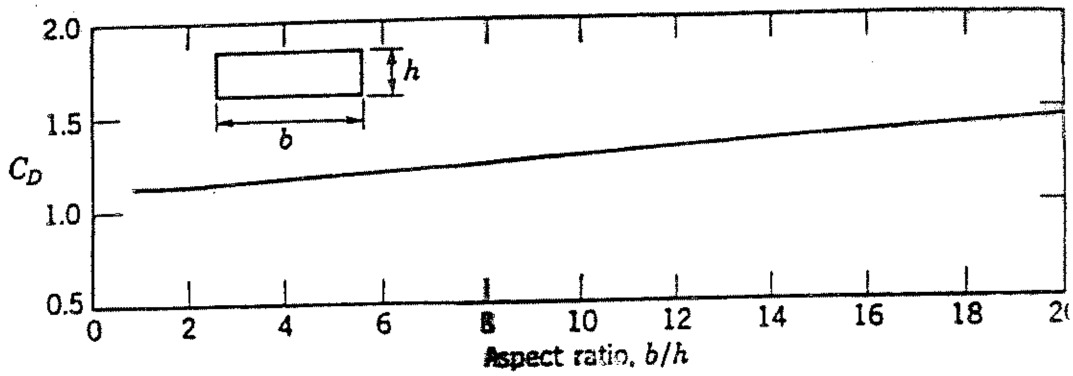
Drag coefficients for bodies of revolution as a function of Reynolds Number (N_R)

(from B.S. Massey, Mechanics of Fluids, 6th ed. Van Nostrand Reinhold, 1989)

log - log

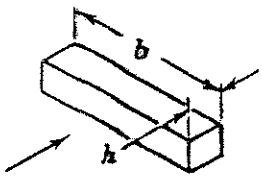
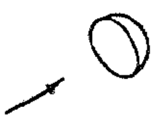
σφαίρα

$\ln C_d = \ln 24 - \ln Re$



Variation of drag coefficient with aspect ratio for a flat plate of finit width normal to the flow with $Re_h > 1000$

Drag Coefficient Data for Selected Objects ($Re \gtrsim 1000$)

Object	Diagram		$C_D(Re \gtrsim 10^3)$
Square cylinder		$b/h = \infty$ $b/h = 1$	2.05 1.05
Disk			1.17
Ring			1.20[b]
Hemisphere (open end facing flow)			1.42
Hemisphere (open end facing downstream)			0.38

(From: R.W Fox + A.T. McDonald, Intro to Fluid Mech. 3rd Ed, J. Wiley+Sons, 1985.)

Διατήρηση Ενέργειας

$$\frac{DE}{Dt} = \frac{DQ}{Dt} - \frac{DW}{Dt} \qquad \frac{DQ}{Dt} = -\oint_S \vec{q}\cdot d\vec{S}$$

$\frac{DE}{Dt}$ — ολική ενέργεια

$\frac{DQ}{Dt}$ — θερμότητα στο σύστημα

$\frac{DW}{Dt}$ — έργο από το σύστημα

$$E = \int_V \rho \varepsilon \, dV \qquad \varepsilon = \varepsilon_0 + \frac{1}{2}u^2 + U$$

ε — πυκνότητα ενέργειας

ε_0 — εσωτερική, $\frac{1}{2}u^2$ — κινητική, U — δυναμική

$\varepsilon_0(\rho, T)$ ορίζεται από την κατάσταση του συστήματος

$P(\rho, T)$ — καταστατική σχέση

Ιδανικό αέριο $\quad \varepsilon_0 = C_v T \qquad P = \rho R T$

$\vec{q}$ — ιδιότητα μεταφοράς. Για μικρή $\vec{\nabla} T$

$$\boxed{\vec{q} = -\kappa \vec{\nabla} T \quad \text{Νομος Fourier}} \qquad q \sim \frac{\text{ενέργεια}}{T \cdot A}$$

$$\frac{DW}{Dt} = -\sum_{ij} \oint_S u_i \overbrace{\sigma_{ji}}^{-P\delta_{ij}+\tau_{ij}} dA_j = \oint_S (P\vec{u})\cdot d\vec{S} - \sum_{ij}\oint u_i \tau_{ji} dA_j$$

$dF_i = \sum_j \sigma_{ji}\, dA_j$ — i-συνιστώσα δύναμης στην dA_j προβολή της επιφάνειας

$\sum u_i dF_i \equiv \vec{u}\cdot d\vec{F}$ — ρυθμός παραγωγής έργου

$$\frac{DE}{Dt} = \int_V \frac{\partial(\rho\varepsilon)}{\partial t} dV + \oint_S (\rho\varepsilon\vec{u}) \cdot d\vec{S}$$

μετατροπή σε ολοκλήρωμα όγκου

$$\int_V \vec{\nabla} \cdot (\rho\varepsilon\vec{u})\, dV$$

$$\oint \underbrace{(\overline{\overline{\tau}} \cdot \vec{u})}_{\text{διάνυσμα}} \cdot d\vec{S} \equiv \sum_j \oint (\overline{\overline{\tau}} \cdot \vec{u})_j \cdot dA_j = \sum_{i,j} \oint (\tau_{ji} u_i)\, dA_j$$

$$\Rightarrow \int \vec{\nabla} \cdot (\overline{\overline{\tau}} \cdot \vec{u})\, dV = \sum_{i,j} \int_V \frac{\partial}{\partial x_j} (\tau_{ji} u_i)\, dV$$

$$= \sum_{i,j} \int_V \left(\frac{\partial u_i}{\partial x_j} \tau_{ji} + u_i \frac{\partial}{\partial x_j} \tau_{ji} \right) dV$$

$$= \int_V \Phi\, dV + \int \vec{u} \cdot (\vec{\nabla} \cdot \overline{\overline{\tau}})\, dV$$

συνάρτηση ιξώδους παραγωγή θερμότητας

$\Phi = 0$

$$\frac{\partial(\rho\varepsilon)}{\partial t} + \vec{\nabla} \cdot (\rho\varepsilon\vec{u}) + \vec{\nabla} \cdot \vec{q} + \vec{\nabla} \cdot (P\vec{u}) + \vec{u} \cdot (\vec{\nabla} \cdot \overline{\overline{\tau}})$$

$$\rho \frac{D\varepsilon}{Dt} = -\vec{\nabla} \cdot \vec{q} - (\vec{u} \cdot \vec{\nabla}) P - P \vec{\nabla} \cdot \vec{u} + \vec{u} \cdot (\vec{\nabla} \cdot \overline{\overline{\tau}}) + \Phi$$

εξίσωση Μηχανικής ενέργειας

$$\rho \frac{D\vec{u}}{Dt} = -\vec{\nabla} P - \rho \vec{\nabla} U + \vec{\nabla} \cdot \overline{\overline{\tau}}$$

$\vec{u}\cdot\{\text{Navier Stokes}\} \Longrightarrow$

$$\frac{1}{2}\rho\frac{Du^2}{Dt} = +\vec{u}\cdot\left(-\vec{\nabla}P + \vec{\nabla}\cdot\bar{\bar{\tau}}\right) - \rho\vec{u}\cdot\vec{\nabla}U$$

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$$\boxed{\rho\frac{D}{Dt}\left(\frac{1}{2}u^2 + U\right) = \vec{u}\cdot\left(-\vec{\nabla}P + \vec{\nabla}\cdot\bar{\bar{\tau}}\right)}$$

διότι $\dfrac{DU}{Dt} = \cancel{\dfrac{\partial U}{\partial t}}^{0} + \vec{u}\cdot\vec{\nabla}U$

$$\varepsilon = \varepsilon_0 + \frac{1}{2}u^2 + U \Longrightarrow$$

$$\boxed{\rho\frac{D\varepsilon_0}{Dt} = -P\vec{\nabla}\cdot\vec{u} - \vec{\nabla}\cdot\vec{q} + \Phi}$$

$\varepsilon_0 = C_v T \quad , \quad \vec{q} = -\kappa\vec{\nabla}T$

$$\rho C_v\frac{DT}{Dt} = -P\vec{\nabla}\cdot\vec{u} + \kappa\nabla^2 T + \Phi \Rightarrow T(\vec{r},t)$$

$$\underbrace{\vec{u},\ P,\ T,\ \rho}_{6} \Leftarrow \left.\begin{array}{l} \vec{\nabla}\cdot\vec{u} = 0 \\ \text{Navier-Stokes} \\ \text{Εσωτερική ενέργεια} \\ P(\rho, T) \end{array}\right\} 6$$

οριακές συνθήκες γιά

$T_1 = T_2$ σε ενδοεπιφάνεια

$\dfrac{\partial T}{\partial n} = 0$ γιά αδιαβατική ροή στην επιφάνεια