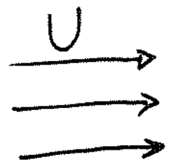
Απλές Ροές Δυναμικού

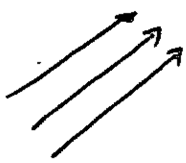


1-D

$$\phi = Ux + \phi_0 \qquad u_x = \frac{\partial \phi}{\partial x} = U$$

$$\psi = Uy + \phi_0 \qquad u_x = \frac{\partial \psi}{\partial y} = U$$

$\vec{u} = (U, 0, 0)$



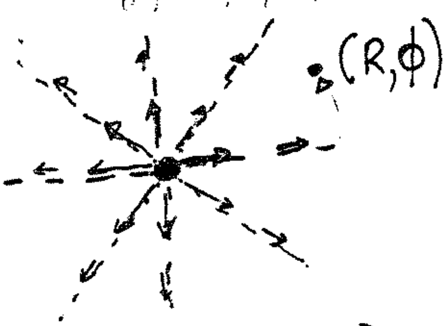
2-D

$$\phi = Ux + Vy + \varphi_0$$

$$\psi = Uy - Vx + \varphi_0$$

$\vec{u} = (U, V, 0)$

Ευθύγραμμη πηγή



$$\nabla^2 \phi = 0 \qquad R \neq 0$$

$$\phi = \frac{Q_2}{2\pi} \ln R \Rightarrow u_R = \frac{Q_2}{2\pi R} \Rightarrow \psi = \frac{Q_2 \phi}{2\pi}$$

$$u_R = \frac{1}{R}\frac{\partial \psi}{\partial \phi}$$

Πηγή στο (x_0, y_0)

$$\phi(x,y) = \frac{Q_2}{2\pi} \ln \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

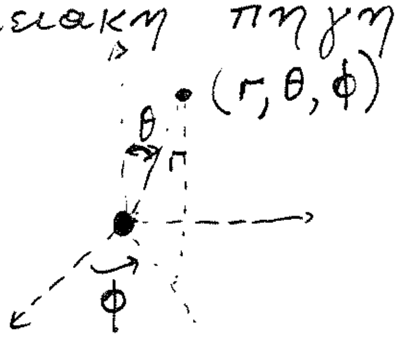
$$\oint \vec{u}_R \cdot d\vec{S} = \underbrace{1}_{\text{ύψος}} \cdot \int_0^{2\pi} u_R \cdot R\, d\theta = Q_2$$

κύλινδρο μοναδιαίου ύψους

Q_2: ρυθμός ροής όγκου/μονάδα ύψους

$\vec{u} = (u_R(R), 0, 0)$

Σημειακή πηγή



$$\nabla^2 \phi = 0 \qquad r \neq 0$$

$$\phi = -\frac{Q}{4\pi r} \Rightarrow u_r = \frac{Q}{4\pi r^2} \Rightarrow$$

$$\oint \vec{u} \cdot d\vec{S} = \int u_r \cdot r^2 d\Omega = Q$$

σφαίρα

$\vec{u} = (u_r(r), 0, 0)$

Θεώρημα Helmholtz.

$$\left.\begin{aligned}\vec{\nabla}\cdot\vec{u} &= \Delta \\ \vec{\nabla}\times\vec{u} &= \vec{\zeta}\end{aligned}\right\} \text{ γνωστά}$$

$$\vec{u} = \vec{u}_e + \vec{u}_v + \vec{u}_0$$

$\vec{\nabla}\cdot\vec{u}_e = \Delta$

$\vec{\nabla}\times\vec{u}_e = 0$

Ροή λόγω διόγκωσης

I

$\vec{\nabla}\cdot\vec{u}_v = 0$

$\vec{\nabla}\times\vec{u}_v = \vec{\zeta}$

Ροή λόγω στροβιλισμού

II

$\vec{\nabla}\cdot\vec{u}_0 = 0$

$\nabla\times\vec{u}_0 = 0$

Ροή με $\Delta=0$, $\vec{\zeta}=$

(οφείλεται σε

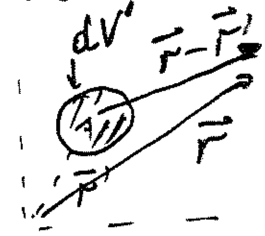
οριακές συνθ.)

III

I) Ροή με διόγκωση

$\vec{\nabla}\times\vec{u}_e = 0 \qquad \vec{u}_e = \vec{\nabla}\Phi_e$

$\vec{\nabla}\cdot\vec{u}_e = \Delta \qquad \nabla^2\Phi_e = \Delta$ – εξίσωση Poisson.

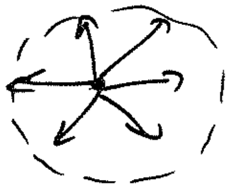


3-D $$\Phi_e(\vec{r}) = -\frac{1}{4\pi}\int_V d^3r' \, \frac{\Delta(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\vec{u}_e(\vec{r}) = \frac{1}{4\pi}\iint d^3r' \, \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\,\Delta(\vec{r}')$$

$$\vec{\nabla}_{\vec{r}}\frac{1}{|\vec{r}-\vec{r}'|} = \vec{\nabla}_{\vec{r}}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} = -\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

Σημειακή πηγή (3-D)



$$\int dV\,\vec{\nabla}\cdot\vec{u} = \int \vec{u}\cdot d\vec{S}^{\;u_r r^2 d\Omega} =$$

$$\Delta(\vec{r}') \rightarrow Q\,\delta(\vec{r}')$$

$$\nabla^2\Phi_e = \Delta(\vec{r}) \Rightarrow \Phi_e = -\frac{1}{4\pi}\int dV' \frac{Q\,\delta(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\Phi_e = -\frac{Q}{4\pi r} \Rightarrow u_r = \frac{Q}{4\pi r^2}$$

Όμως $\vec{\nabla}\cdot\vec{u}=0$ παντού πλήν $r=0$.

Συνάρτηση ροής $\psi(r,\theta) = -\frac{Q}{4\pi}\cos\theta$

2-D

$$\Phi_e(\vec{R}) = \frac{1}{2\pi}\iint_S dS' \ln|\vec{R}-\vec{R}'|\;\Delta(\vec{R}')$$

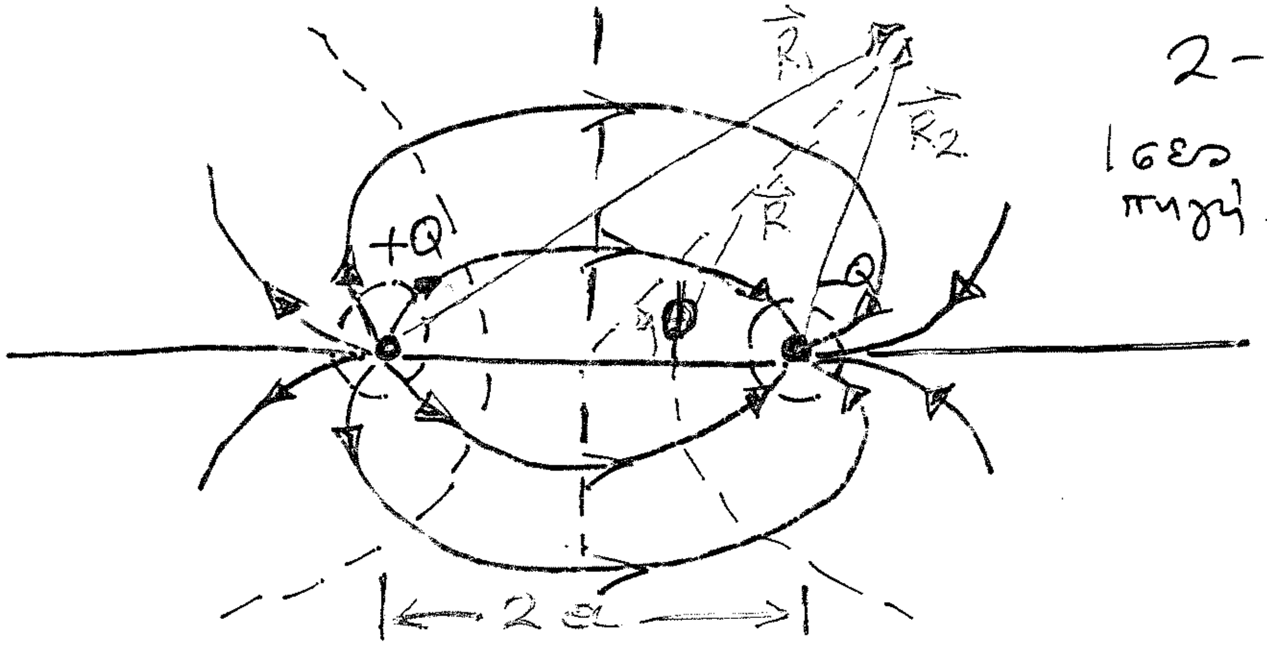
$$\vec{u}_e(\vec{R}) = \frac{1}{2\pi}\iint_S dS' \frac{\vec{R}-\vec{R}'}{|\vec{R}'-\vec{R}'|}\,\Delta(\vec{R}')$$

Ευθύγραμμη πηγή $\Delta(\vec{R}') = Q_2\,\delta(\vec{R}')$

$$\Phi_e(\vec{R}) = \frac{Q_2}{2\pi}\ln R$$

$$u_R(R) = \frac{Q_2}{2\pi R}$$

Συνάρτηση ροής $\psi(R,\phi) = \frac{Q_2}{2\pi}\phi$



2-D
Ισες αντίθε πηγής

$$\Phi_e = +\frac{Q}{2\pi}\left[\ln R_1 - \ln R_2\right] = +\frac{Q}{2\pi}\ln\frac{R_1}{R_2}$$

$$R_{1,2} = R\sqrt{1 \pm \frac{2a}{R}\cos\phi + \left(\frac{a}{R}\right)^2} \qquad \vec{R}_{1,2} = \vec{R} \pm \vec{a}$$

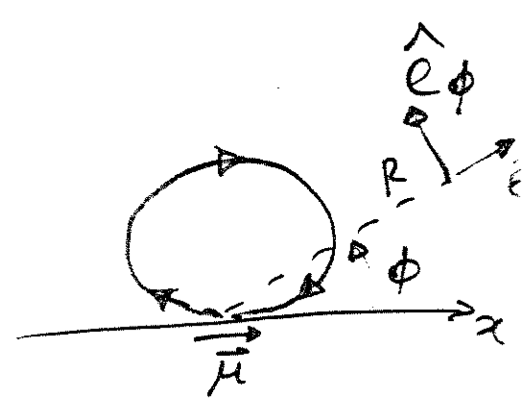
$R \gg a$ ή $a \to 0$ αλλά $2aQ \to \mu$

$$\Phi_e = +\frac{\mu\cos\phi}{2\pi R} \Rightarrow -\frac{\vec{\mu}}{2\pi}\cdot\vec{\nabla}\left[\ln R\right]$$

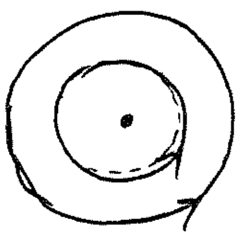
$$\Psi_e = \frac{\mu\sin\phi}{2\pi R}$$

$$u_R = \frac{\partial\Phi_e}{\partial R} = -\frac{\mu\cos\phi}{2\pi R^2}$$

$$u_\phi = \frac{1}{R}\frac{\partial\Phi_e}{\partial\phi} = -\frac{\mu\sin\phi}{2\pi R^2}$$



Ευθύγραμμος στρόβιλος $\quad \vec\nabla\cdot\vec u = 0 \quad \Rightarrow u_\phi(R)$

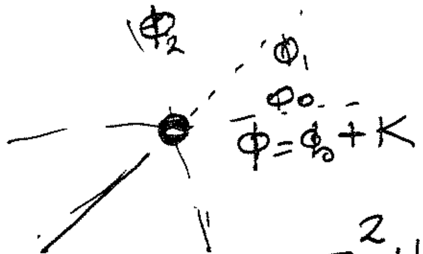


$$\vec\nabla\times\vec u = 0$$

$$\frac{1}{R}\frac{\partial}{\partial R}(R u_\phi) = 0 \Rightarrow u_\phi \sim \frac{1}{R}$$

or

$\vec u = (0, u_\phi(R), 0) \qquad \nabla^2\Phi = 0 \qquad R \neq 0$



$$\Phi = \frac{K}{2\pi}\phi + \varphi_0 \to u_\phi = \frac{1}{R}\frac{\partial\Phi}{\partial\phi} = \frac{K}{2\pi R}$$

$$\nabla_2^2\psi = \zeta_z = \frac{K}{2\pi}\delta(R)$$

$$\nabla_2^2\psi = \left\{\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2}{\partial\phi^2}\right\}\psi = \delta(R)$$

$$\psi = -\frac{K}{2\pi}\ln R$$

$$u_\phi = -\frac{\partial\psi}{\partial R} = \frac{K}{2\pi R}$$

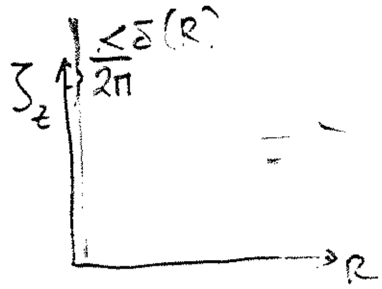
κυκλοφορία

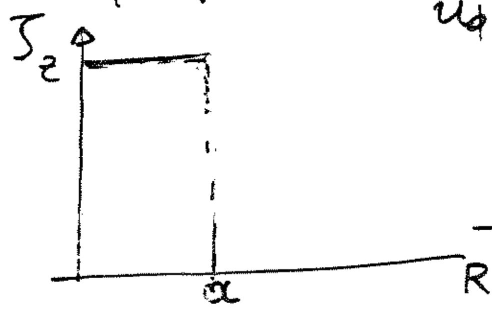
$$\oint \vec u_\phi\cdot d\vec l = \int_0^{2\pi} u_\phi R\, d\phi = K$$

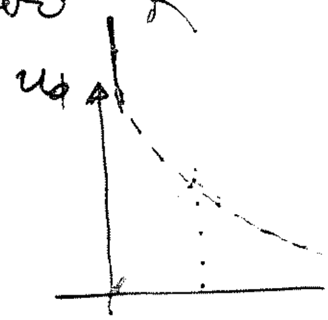
$$\psi(R+\delta R) - \psi(R) = \int_R^{R+\delta R} u_\phi \cdot dR = \frac{K}{2\pi}\ln\frac{R+\delta R}{R}$$

$\oint \vec u\cdot d\vec l = K$ ανεξάρτητο του R

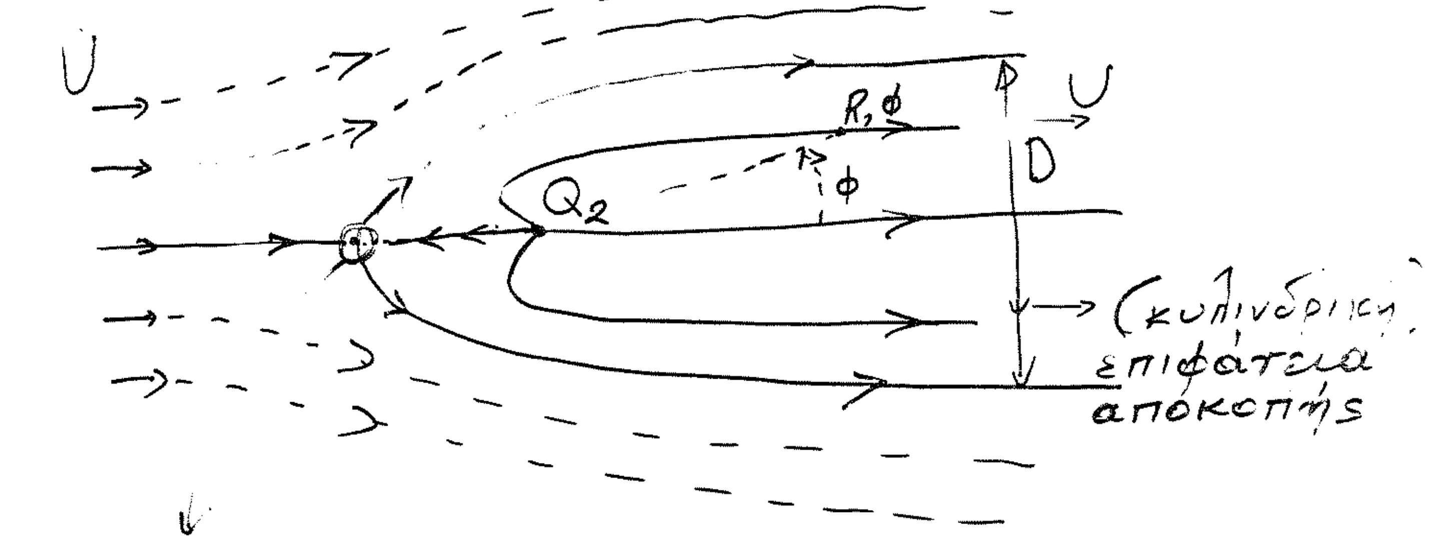
κύκλος ακτίνας R.







Σταθερή ροή + πηγή (2-D)



Σημείο ηρεμίας $\vec{u}=0$

Μόνο σε σημεία ηρεμίας διασταύρωση γραμμών.

Υπέρθεση με οριακές συνθήκες κυλινδρικές

$$\Phi = Ux + \frac{Q}{2\pi}\ln\sqrt{x^2+y^2} \equiv U R\cos\phi + \frac{Q}{2\pi}\ln R$$

$$u = \frac{\partial \Phi}{\partial x} = U + \frac{Q}{2\pi}\frac{x}{x^2+y^2}$$

$$v = \frac{\partial \Phi}{\partial y} = \frac{Q}{2\pi}\frac{y}{x^2+y^2}$$

$v=0 \Rightarrow y_0=0$

$u|_{y=0} = U + \frac{Q}{2\pi x} \stackrel{?}{=} 0$

$(x_0, 0)$ - σημείο ηρεμίας

$x_0 = -\frac{Q}{2\pi U}$

$x \to \infty \quad \vec{u} \to (U, 0)$

άνοιγμα επιφάνειας

$DU = Q_2 \Rightarrow D = \frac{Q_2}{U}$

ροή πηγής εντός επιφάνειας

$$u_R = \frac{\partial \Phi}{\partial R} = U\cos\phi + \frac{Q}{2\pi R}$$

$$u_\phi = \frac{1}{R}\frac{\partial \Phi}{\partial \phi} = -U\sin\phi$$

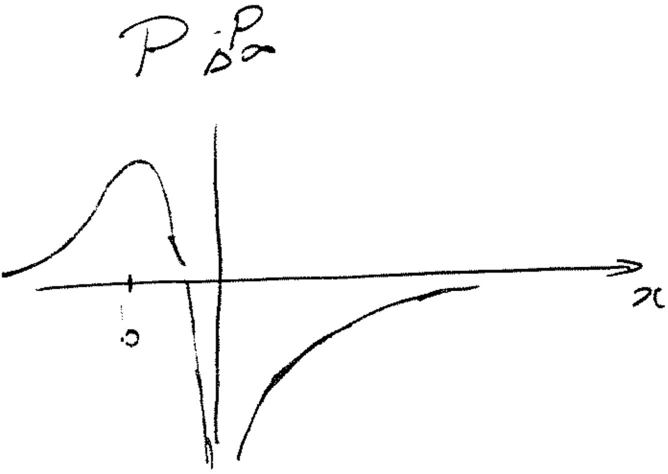
σημείο ηρεμίας $R_0, \phi_0 \Rightarrow \left(\frac{Q}{2\pi U}, \pi\right)$

$$u_R^2 + u_\phi^2 = U^2 + \frac{Q^2}{4\pi R^2} + 2U\cos\phi\,\frac{Q}{2\pi R}$$

$$\frac{1}{2}u^2 + \frac{P}{\rho} = \frac{1}{2}U^2 + \frac{P_\infty}{\rho}$$

$$P - P_\infty = -\frac{1}{2}\rho\left\{\frac{Q^2}{4\pi R^2} + 2U\cos\phi\,\frac{Q}{2\pi R}\right\}$$

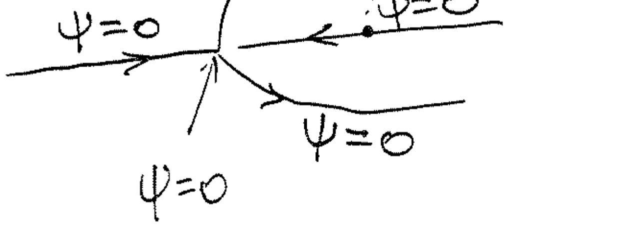
Προσοχή!! Όχι υπέρβαση στην πίεση



$\text{o} = \frac{Q}{2\pi U}$

επιφάνεια αποκοπής

$$\psi = UR\sin\phi + \frac{Q}{2\pi}(\phi - \pi) \qquad \psi(\phi=\pi) = 0$$



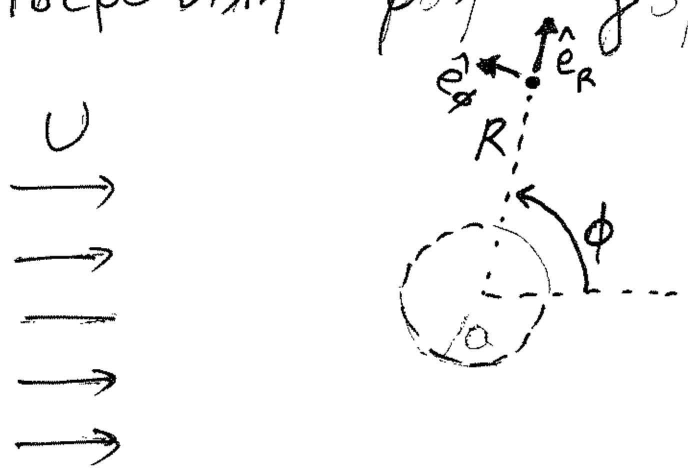
εξίσωση

$$R = \frac{Q(\pi - \phi)}{2\pi U\sin\phi}$$

$\phi = 63°$ $\quad V_{max} = 1.26U$

ελάχιστη P

Αστρόβιλη ροή γύρω από κύλινδρο



$\vec{u}(R,\phi) = u_R \hat{e}_R + u_\phi \hat{e}_\phi$

$$\left.\begin{aligned} \vec{u}(R\to\infty) &= U\hat{i} \\ u_R(R=a,\phi) &= 0 \end{aligned}\right\} B.C$$

$\vec{u} = -\vec{\nabla}\Phi \quad , \Phi(R,\phi)$

$$\nabla^2\phi = 0 \Rightarrow \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

$$\left.\begin{aligned} \Phi(R\to\infty,\phi) &= -Ux = -UR\cos\phi \\ \left.\frac{\partial\Phi}{\partial R}\right|_{R=a} &= 0 \end{aligned}\right\} B.C$$

Γενική λύση Laplace (2-D)

$$\Phi(R,\phi) = \mathcal{R}(R)\,Q(\phi)$$

$$\frac{R}{\mathcal{R}}\frac{d}{dR}\left(R\frac{d\mathcal{R}}{dR}\right) = -\frac{1}{Q}\frac{d^2Q}{d\phi^2} = m^2$$

$$\left.\begin{aligned} Q_m(\phi) &= E\cos m\phi + D\sin m\phi \\ \mathcal{R}_m(R) &= AR^m + BR^{-m} \end{aligned}\right\} m = 1, 2, \ldots \qquad m \neq 0$$

$$\Phi_0(R,\phi) = \mathcal{R}_0(R)\,Q(\phi) = (A_0 \ln R + B_0)(C_0 + D_0\phi) \quad m=0$$

Για γενικές οριακές συνθήκες

$$\Phi(R,\phi) = \Phi_0(R,\phi) + \sum_{m\geq 1}^{\infty} \mathcal{R}_m(R)\,Q_m(\phi)$$

Οριακή συνθήκη γιά $R \to \infty \Rightarrow$

$$\Phi(R,\phi) = f(R)\cos\phi \quad , \quad \left.\frac{\partial f}{\partial R}\right|_{R=a} = 0$$

$m = 1$ $\Downarrow$

$$f(R) = \left(AR + \frac{B}{R}\right) \Rightarrow$$

$$f(R \to \infty) \to -UR = AR \Rightarrow \quad A = -U$$

$$\left.\frac{\partial f}{\partial R}\right|_{R=a} = A - \left.\frac{B}{R^2}\right|_{R=a} = 0 \Rightarrow \quad B = Aa^2 = -Ua^2$$

$$\Phi(R,\phi) = -U\left(R + \frac{a^2}{R}\right)\cos\phi$$

$$u_R(R,\phi) = -\frac{\partial \Phi}{\partial R} = U\left(1 - \frac{a^2}{R^2}\right)\cos\phi$$

$$u_\phi(R,\phi) = -\frac{1}{R}\frac{\partial \Phi}{\partial \phi} = -U\left(1 + \frac{a^2}{R^2}\right)\sin\phi$$

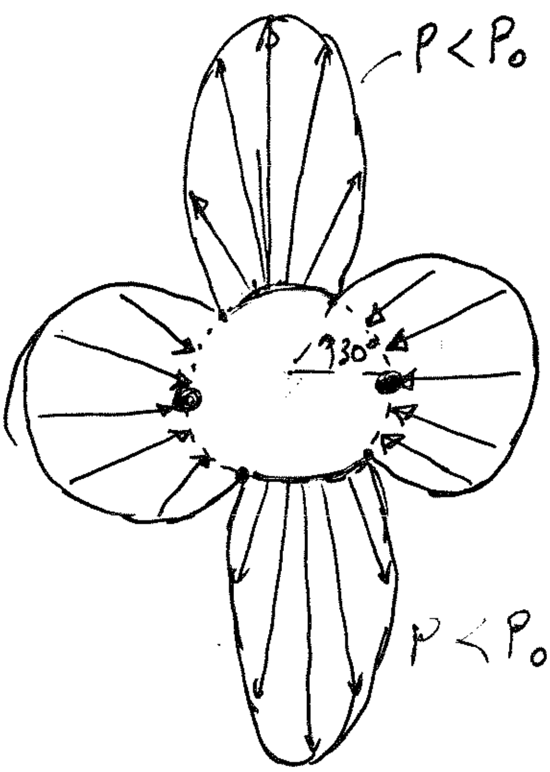
$$u^2 = U^2\left(1 + \frac{a^4}{R^4} - 2\frac{a^2}{R^2}\cos 2\phi\right)$$

$$P - P_0 = \frac{1}{2}\rho U^2 - \frac{1}{2}\rho u^2 = -\frac{1}{2}\rho U^2\left(\frac{a^4}{R^4} - 2\frac{a^2}{R^2}\cos 2\phi\right)$$

Συντελεστής πίεσης $C_p = \dfrac{P - P_0}{\frac{1}{2}\rho U^2}$

$R = a \qquad C_p = 1 - 4\sin^2\phi$

$$C_p(R=a) = 0 \quad \Rightarrow \sin\phi = \pm\frac{1}{2} \Rightarrow \phi = \pm\frac{\pi}{6}, \pm\left(\pi - \frac{\pi}{6}\right)$$

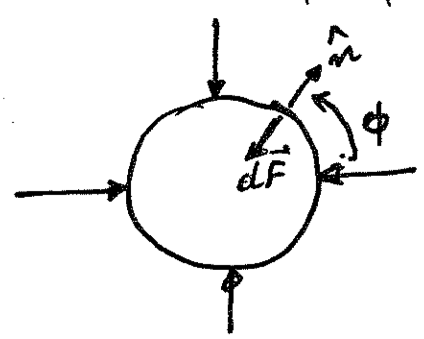


Πιθανή παραμόρφωση σφαίρας

$C_p(R=a, \phi=0,\pi) = 1$

Σημεία ηρεμίας => μέγιστη πίεση.

Δύναμη στον κύλινδρο $\vec{F}=0$ (συμμετρία



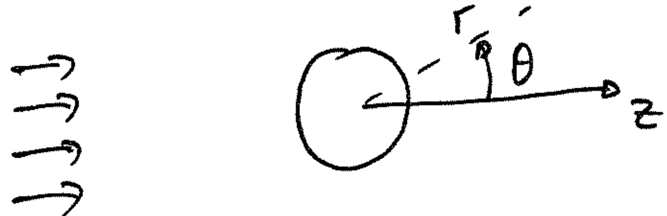
$$\vec{F} = -\oint_S P(R=a,\phi)\,\hat{n}\,dS$$

$$\hat{n} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j} \quad , dS = a\,d\phi$$

Δύναμη=0 γιά οποιαδήποτε διατομή!!

Λόγος => έλλειψη ιξώδους.

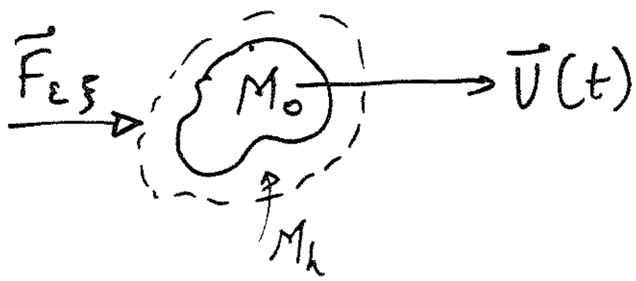
Ροή γύρω από σφαίρα $\Phi(r,\theta) = -Ur\cos\theta\left(1+\frac{a^3}{2r^3}\right)$



Κίνηση σφαίρας σε υγρό

$$\Phi(x,y,\xi=z-Vt) = -\frac{Va^3}{2}\frac{z-Vt}{\left[R^2+(z-Vt)^2\right]^{3/2}}$$

Υδροδυναμική μάζα στερεού.



$$\vec{F}_{\varepsilon\xi} = (M_0+M_h)\frac{d\vec{U}}{dt}$$

$\vec{F}_{\varepsilon\xi}$ = ρυθμό μεταβολής ορμής σώματος + ρευστού

$$M_0\frac{d\vec{U}}{dt} + \underbrace{\rho\int_V \frac{D\vec{u}}{Dt}\,dV}_{M_h\frac{d\vec{U}}{dt}}$$

- σε όλο τον όγκο

Σφαίρα $M_h = \frac{1}{2} M_{\sigma\varphi\alpha\acute{\iota}\rho\alpha\varsigma}$.

Άλλο σώμα $M_h = K M_{\sigma\acute{\omega}\mu\alpha\tau o\varsigma}$.

K - εξαρτάται από προσανατολισμό

Δύναμη στο σώμα $\vec{F}_{\varepsilon\xi} - M_h\frac{d\vec{V}}{dt}$

Αν $\vec{V}$ = σταθερή τότε δεν έχουμε δύναμη αντίδρασης. ⇒ έλλειψη ιξώδους.

Μη συμμετρικό σώμα $(F_{\alpha\nu})_i = (M_h)_{ij}\frac{dU_j}{dt}$